ANALOG OF THE TAYLOR–PROUDMAN THEOREM IN FLOWS WITH VELOCITY SHEAR

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UDC 532.527

A number of properties of small perturbations of a fluid experiencing solid-body rotation are widely known that can be classified as a manifestation of unusual flow "elasticity." Such properties are the wave nature of the development of perturbations and the phenomenon of forming "Taylor columns." The latter result is called the Taylor– Proudman theorem, and states that in slow motion of a solid body in a rotating fluid the flow is two-dimensional to a first approximation (is independent of the coordinate z along the axis of rotation). The existence of this kind of flow structure, first predicted theoretically, was later confirmed in a number of experiments [1]. Both the properties of the wave motions [1, 2] and phenomena of the Taylor column type are evidence of the strong anisotropy of the "elastic" properties of the medium. The elasticity is associated with flexure of the vortex lines and is less the less the lines flex. If the field of the perturbations is two-dimensional, so that the motion occurs without bending of vortex lines, then no elasticity appears, and the flow is of the Taylor column type. The cause of these fluid properties is the gyroscopic behavior of the rotating liquid particles (see [3-5]). On the basis of this qualitative picture one can predict that any stable vortex flow possesses elastic properties. However, in general fluid particles are also subject to strains. The latter may play a destabilizing role in a flow, diminishing its elasticity and even leading to instability [5-7]. Examples of how the presence of strains alters the properties of wave motion in stable flows are given in [8, 9]. The question arises: Can one find phenomena of the Taylor column type in vortex flows which differ from the solid-body rotation form? Three examples are constructed below to show that these phenomena are theoretically possible. The method of proof in each case is practically a repeat of the proof of the Taylor–Proudman theorem. All the examples consider a model of an ideal incompressible fluid with constant density.

1. **Flows with Circular Streamlines.** We consider an axisymmetric steady-state flow with circular streamlines. We introduce cylindrical coordinates (φ, r, z) with the z axis directed along the symmetry axis. Only the angular component of the velocity U = U(r) and the axial component of the vorticity $\Omega = dU/dr + U/r$ differ from zero. In this flow let there be an axisymmetric solid body whose projection in the plane z = const is the ring $a < r < b$. This body moves along the z axis with a small constant speed w_0. It is necessary to find the velocity field of the fluid.

We assume that the quantity $w_0/(b - a)\Omega$ is so small that the fields of the perturbations are described by the linearized system of equations

$$\Omega v = 0,\ \frac{2U}{r}u = -\frac{\partial p}{\partial r},\ 0 = \frac{\partial p}{\partial z},\ \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial w}{\partial z} = 0, \tag{1.1}$$

where u, v, and w are components of the velocity field perturbations corresponding to the coordinates φ, r, z; p is the field of pressure perturbations. Because of the problem symmetry the perturbation fields are independent of φ.

We achieve steady-state by converting to a coordinate system fixed in the body. From the first equation in Eq. (1.1) we have that v = 0, from which (after differentiation with respect to z) we have u = u(r), and from the continuity equation we have that w = w(r). By determining w(r) from the boundary conditions, we come to the result that the desired velocity field is given by the inequalities

$$v(r) = 0,\ w(r) = \begin{cases} -w_0 & \text{for } r < a,\quad r > b, \\ 0 & \text{for } a < r < b. \end{cases} \tag{1.2}$$

The function u(r) can be assigned arbitrarily. The case $u(r) \neq 0$ is equivalent to a change in U(r). Thus, we have shown that, as in the case of a flow exhibiting solid-body rotation [1], the slowly moving body presses through the entire column of fluid. We note that the field of the perturbations is not attenuated for $z \to \pm\infty$. If we require that there be attenuation, then the solution will be unsteady, and will correspond to motion with radiating waves. For the solid-body rotation case it has been shown [1] that such unsteady conditions tend to the solutions with the Taylor columns for $t \to \infty$. The tangential discontinuities at r = a, r = b in Eq. (1.2) are of the same type as in inertial boundary layers [1]. The circular flow case is a direct generalization of solid-body rotation of fluid, and it is therefore natural that we have a result similar to the Taylor–Proudman theorem.

2. **Plane-Parallel Flow.** A more unexpected result is that similar results can be obtained for plane parallel flow. We shall examine such a flow. We introduce rectangular coordinates x, y, z with the x axis directed along the flow, such that

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 40-43, November-December, 1980. Original article submitted October 15, 1979.

 0021–8944/80/2106-0772$07.50

the unique non-zero component of velocity depends only on y: $U \doteq U(y)$. Let there be a cylindrical body in this flow such that the generator of the cylinder is parallel to the x axis. This body moves along the z axis with a small constant velocity w_0. It is necessary to find the velocity field of the fluid.

The argument proceeds along the same lines as in the previous example. Because of the problem symmetry the fields of the perturbations are independent of x. The linearized equations of motion give

$$U'v = 0,\ \partial p/\partial y = 0,\ \partial p/\partial z = 0,\ \partial v/\partial y + \partial w/\partial z = 0.$$

Here u, v, and w are the components of the field of velocity perturbations corresponding to the coordinates x, y, and z. From the first equation we have v = 0, from the second and third we have p = 0, and from the fourth we have that w = w(y). Let the projection of the body on the plane (x, y) be the strip $a < y < b$. By satisfying the boundary conditions we obtain that the velocity field (in the coordinate system fixed in the cylinder) is given in the form

$$v(y) = 0,\ w(y) = \begin{cases} -w_0 & \text{for } y < a,\ y > b, \\ 0 & \text{for } a < y < b. \end{cases}$$

The function u(y, z) can be given arbitrarily.

Thus, the velocity field in the problem must be two-dimensional, so that in its motion the body presses through the entire layer of fluid $a < y < b$.

3. **Axisymmetric Flow with Straight Streamlines.** In examples 1 and 2 the vortex lines of the main flow were straight. It is shown in this example that similar results can be obtained also for flows with curved vortex lines.

We consider axisymmetric flow with straight streamlines (e.g., flow in a tube). Let z be the flow symmetry axis. In the cylindrical coordinate system (φ, r, z) only the z velocity component W = W(r) is nonzero. This flow has a solid strip ($a < r < b,\ \varphi = \varphi_0,\ -\infty < z < +\infty$), which rotates slowly around the z axis at constant angular velocity ω_0, such that $\varphi_0 = \omega_0 t$. We require to find the velocity field corresponding to this flow problem.

In a coordinate system rotating with velocity ω_0 the fields of the perturbations do not depend on z and t. After linearization of the equations of motion, written in this system, by a method similar to that used above, we obtain the conditions v = 0, u = u(r); the condition w = w(r, φ) can be assigned arbitrarily. The notation for the components of the velocity perturbations are the same as in example 1. By satisfying the boundary conditions we obtain the result that the cylindrical volume of fluid $a < r < b$ (a ring) rotates in solid-body fashion along with the strip.

We note that in all the examples nothing has been said about flow stability. The problem of stability of these flows of an ideal fluid is complex, and for the main part there are as yet no solutions [5, 10, 11].

The examples presented illustrate the uniqueness of the gyroscopic properties of vortex flows, and the conditions for which these properties, even with shear layers present, can generate phenomena which are consonant with Taylor columns.

In a real viscous fluid the presence of shear layers in the flow, together with the adhesion conditions at the body boundaries, will introduce additional perturbations into the stream. Therefore an experimental verification of the flows considered will encounter considerable difficulty.

The mathematical formulation of the results obtained is similar. In the examples listed there are not smooth steady-state solutions to the flow problems in the linear approximation. The solutions here are discontinuous and are of the Taylor column type.

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